A stable RBF method for nonlinear sinh-Gordon equation in two dimensions

Mohammad Navaz Rasoulizadeh[1,*] and Omid Nikan[2]

[1] Department of Mathematics, Velayat University, Iranshahr, Iran; mn.rasoulizadeh@velayat.ac.ir
[2]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran; omidnikan77@yahoo.com

Abstract. This paper presents a locally stabilized radial basis functions (RBF) meshless method based on the QR algorithm for finding the approximate solutions of the two dimensional nonlinear sinh-Gordon (ShG) equation. The proposed method consists of two phases. First, a semi-discrete approach with second order accuracy is performed by means of the central finite difference (FD) and θ-weighted methods. Second, a local RBF based on a FD is employed to discretize the space variables. The QR algorithm is used for numerically stable computations with RBFs for all values of the free shape parameter. Numerical example confirms the feasibility of the proposed method.

1. Introduction

In this paper, we consider the two dimensional sinh-Gordon equation [1, 2] as:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \Delta u(x,y,t) + \sinh(u(x,y,t)) = f(x,y,t),\ (x,y) \in \Omega \subseteq \mathbb{R}^2,\ 0 < t \le T, \tag{1a}$$

Th initial conditions (ICs) and the boundary condition (BC) are given as

$$u(x,y,0) = g_1(x,y), \qquad (x,y) \in \Omega \cup \partial\Omega, \tag{1b}$$

$$\frac{\partial u(x,y,t)}{\partial t}|_{t=0} = g_2(x,y), \qquad (x,y) \in \Omega, \tag{1c}$$

$$u(x,y,t) = \Psi(x,y,t), \quad (x,y) \in \partial\Omega,\ 0 < t \le T, \tag{1d}$$

in which functions f, g_1, g_2 and Ψ are given and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

In this paper, we present a locally stabilized radial basis functions (RBF) meshless method based on the QR algorithm for finding the approximate solutions of the two dimensional nonlinear sinh-Gordon (ShG) equation. The proposed method includes of two phases. First, a semi-discrete approach with second order accuracy is performed by means of the central finite difference (FD) and θ-weighted methods. Second, a local RBF based on a FD is employed to discretize the space variables.

*Speaker

2. Numerical Method

2.1. Time discretization. First, $m+1$ distinct points $t_j = jk, j = 0, 1, \cdots, m$ with time step k are selected. Then, the central FD and θ-weighted ($0 < \theta \le \frac{1}{2}$) methods are adopted over three consecutive temporal steps t_{j-1}, t_j , t_{j+1} on Eq.(1a) as

$$\frac{u^{j-1} - 2\,u^j + u^{j+1}}{k^2} - \Big(\theta\Delta u^{j+1} + (1-2\theta)\Delta u^j + \theta\Delta u^{j-1}\Big) + \sinh(u^j) = f(x,y,t_j), \quad for\ \underline{x} = (x,y) \in \Omega, \tag{2a}$$

$$u^j = \Psi(x,y,t_j), \quad for\ \underline{x} = (x,y) \in \partial\Omega, \tag{2b}$$

where $j = 0, 1, \cdots, (m-1)$ and $u^j = u(\underline{x}, t_j) = u(x, y, t_j)$.
Eq. (2a) can be rewritten as:

$$\Big(1 - \theta\, k^2\,\Delta\Big)u^{j+1} = \Big(2 + (1-2\theta)\,k^2\,\Delta\Big)u^j - \Big(1 - \theta\,k^2\,\Delta\Big)u^{j-1} - k^2\,\sinh(u^j) + k^2\,f(x,y,t_j), \quad for\ \underline{x} = (x,y) \in \Omega. \tag{3}$$

2.2. Space discretization. Suppose that $X = \{\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N\}$ as N distinct collocation points are chosen in domain $\Omega \cup \partial\Omega$ in which $\underline{x}_i = (x_i, y_i)$, $i = 1, \cdots, N_1$ are the N_1 internal points and $\underline{x}_i, i = N_1+1, \cdots, N$ are the N_2 boundary nodes.
For each point $\underline{x}_i$, stencil $I_i = \{\underline{x}_j \in X : \|x_j - x_i\| \le R\} = \{\underline{x}_{i_1}, \underline{x}_{i_2}, \cdots, \underline{x}_{i_{n_i}}\}$ in support radius R (Figure 1) is chosen. We assume $\underline{x}_i = \underline{x}_{i_1}$ ($i = i_1$) without loss of generality. For every point $\underline{x}_i$ and its stencil I_i, the weights $\underline{w}_{xx,i} = [w_{xx,i1}, \cdots, w_{xx,in_i}]^T$ and $\underline{w}_{yy,i} = [w_{yy,i1}, \cdots, w_{yy,in_i}]^T$ corresponding to $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ will be computed with RBF-QR-FD method. Thus the weights $\underline{w}_i = [w_{i1}, \cdots, w_{in_i}]^T$ for Δ can be achieved by summing up $\underline{w}_{xx,i}$ and $\underline{w}_{yy,i}$ as

$$\Delta u^v(\underline{x}_i) = \sum_{s=1}^{n_i} w_{is}\, u^v_{i_s}, \quad i = 1, \cdots, N_1,\ v = j-1, j, j+1, \tag{4}$$

where $u^v_{i_s} = u(\underline{x}_{i_s}, t_v)$.

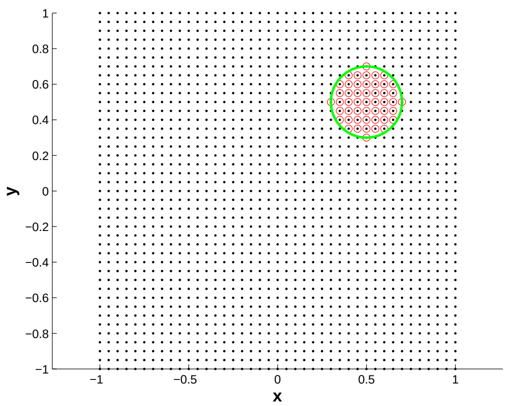

FIGURE 1. A schematic of a stencil for uniform grid nodes with $\delta x = \delta y = 0.05$ at center node $\underline{x}_i = (0.5, 0.5)$ and $R = 0.2$.

The weights w_{is} are only dependent on stencil nodes. Applying the collocation methods on interior nodes in Eq. (3) and using Eq. (4) concluded that

$$
\begin{aligned}
u_i^{j+1} - \theta\, k^2 \Big(\sum_{s=1}^{n_i} w_{is}\, u_{i_s}^{j+1}\Big) &= 2u_i^j + (1-2\theta)\, k^2 \Big(\sum_{s=1}^{n_i} w_{is}\, u_{i_s}^{j}\Big) \\
&- \Bigg(u_i^{j-1} - \theta\, k^2 \Big(\sum_{s=1}^{n_i} w_{is}\, u_{i_s}^{j-1}\Big)\Bigg) - k^2 \sinh(u_i^j) + k^2\, f(x_i, y_i, t_j).
\end{aligned} \tag{5}
$$

Eq.(5) and Eq.(2b) lead to the following $N \times N$ system:

$$
\begin{aligned}
&(1-\theta\, k^2\, w_{i1})u_i^{j+1} - (\theta\, k^2\, w_{i2})u_{i_2}^{j+1} - \cdots - (\theta\, k^2\, w_{in_i})u_{i_{n_i}}^{j+1} = \\
&\Big(2 + (1-2\theta)\, k^2\, w_{i1}\Big)u_i^j + ((1-2\theta)\, k^2\, w_{i2})u_{i_2}^j + \cdots + ((1-2\theta)\, k^2\, w_{in_i})u_{i_{n_i}}^j \\
&- \Big((1-\theta\, k^2\, w_{i1})u_i^{j-1} - (\theta\, k^2\, w_{i2})u_{i_2}^{j-1} - \cdots - (\theta\, k^2\, w_{in_i})u_{i_{n_i}}^{j-1}\Big) \\
&- k^2\, \sinh(u_i^j) + k^2\, f(x_i, y_i, t_j), \qquad i = 1, \cdots, N_1,
\end{aligned} \tag{6a}
$$

$$
u_i^{j+1} = \Psi(x_i, y_i, t_{j+1}), \qquad i = N_1+1, \cdots, N. \tag{6b}
$$

By partitioning the vector $U^{j+1} = [u_1^{j+1}, u_2^{j+1}, \cdots, u_N^{j+1}]^T = [U_1^{j+1}, U_2^{j+1}]^T$, where $U_1^{j+1} = [u_1^{j+1}, u_2^{j+1}, \cdots, u_{N_1}^{j+1}]^T$ and $U_2^{j+1} = [u_{N_1+1}^{j+1}, \cdots, u_N^{j+1}]^T$ correspond to the internal and boundary nodes, the matrix form of Eq.(6) will be written as:

$$
\begin{aligned}
U_2^{j+1} &= Si^{j+1}, \\
A_1\, U_1^{j+1} &= B_1\, U_1^j - A_1\, U_1^{j-1} - S^j + F^j \\
&\quad + B_2\, U_2^j - A_2\, (U_2^{j+1} + U_2^{j-1}), \quad j = 1, 2, \cdots, (m-1),
\end{aligned} \tag{7}
$$

where sparse matrices $A_{N\times N}$ and $B_{N_1\times N}$ with blocks $A_{N\times N} = \begin{bmatrix} A_1 & A_2 \\ 0 & I_{N_2} \end{bmatrix}$, $B_{N_1\times N} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ correspond to internal and boundary points, and $N_1 \times 1$ known vectors S^j, Si^{j+1} and F^j are as follows:

$$
\begin{aligned}
&A_{ii} = 1 - \theta\, k^2\, w_{i1}, \quad B_{ii} = 2 + (1-2\theta)\, k^2\, w_{i1}, \\
&A_{i\, i_s} = -\theta\, k^2\, w_{is}, \quad B_{i\, i_s} = (1-2\theta)\, k^2\, w_{is}, \\
&S_i^j = k^2\, \sinh(u_i^j), \quad F_i^j = k^2\, f(x_i, y_i, t_j), \\
&i = 1, \cdots, N_1\,, \quad s = 2, \cdots n_i, \quad j = 1, 2, \cdots, (m-1),
\end{aligned} \tag{8}
$$

$$
A_{ii} = 1, \quad Si_i^{j+1} = \Psi(x_i, y_i, t_{j+1}), \quad i = N_1+1, \cdots, N,\ j = 1, 2, \cdots, (m-1).
$$

The other elements of vectors and matrices are equal zero.
When in Eq. (6a) $j = 0$, Eq.(1c) and central FD method at $t = 0$ are used to remove $u_{i_s}^{-1}$. Thus,

$$
u_{i_s}^{-1} = u_{i_s}^1 - 2\, k\, g_2(\underline{x}_{i_s}), \quad i = 1, \cdots, N,\ s = 1, \cdots n_i. \tag{9}
$$

By substituting Eq. (9) into Eq. (6a) for $j = 0$, the matrix form of linear systems at the time level $t_0 = 0$ will be obtained as:

$$
U_2^1 = Si^1,
$$

$$(2A_1)\, U_1^1 = B_1\, U_1^0 + G^0 - \; S^0 + F^0 + B_2\, U_2^0 - (2A_2)\, U_2^1, \tag{10}$$

where $G_i^0 = 2k\,(1-k^2\theta w_{i1})\, g_2(\underline{x}_{i_1}) - 2\theta k^3 \sum_{s=2}^{n_i} w_{is}\, g_2(\underline{x}_{i_s}),\quad i=1,\cdots,N_1$ and the other vectors and matrices are as in Eq. (8) with $j=0$.

3. Numerical illustrations

In this section, we present a numeral example to verify the accuracy and efficiency of the proposed method. The error norms and computational convergence orders are computed as follows:

(1) $L_\infty = \|\underline{U}_e - \underline{U}\|_\infty = \max_{1\le i\le N_1} |\underline{U}_e(\underline{x}_i) - \underline{U}(\underline{x}_i)|$,

(2) $RMS = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} (\underline{U}_e(\underline{x}_i) - \underline{U}(\underline{x}_i))^2 \right)^{\frac{1}{2}}$,

(3) $C-order = \frac{\log\,(\frac{E_j}{E_{j+1}})}{\log\,(\frac{k_j}{k_{j+1}})}$,

where $\underline{x}_i,\ i=1,\cdots,N$ are the collocation nodes, $\underline{U}_e$ and $\underline{U}$ are the exact and computational values of $u(x,y,t)$ respectively and E_j is the L_∞ error respect to k_j or h_j.

The Matlab software and the kdtree package by Guy Shechter [3] for constructing stencils are used to apply RBF-QR-FD method. Moreover, we use $\theta = \frac{1}{12}$, the famous Numerov's method and Gaussian RBFs.

Example 3.1. In this example, the elliptical ring soliton of Eq. (1) is considered. The exact solution is in the following form

$$u(x,y,t) = 4\, \tan^{-1}(\exp(t + \frac{1}{6}\sqrt{36+30x^2+12xy+30y^2}\,)) \qquad (x,y)\in\Omega, \quad t\ge 0.$$

The ICs and BCs as well as function $f(x,y,t)$ are good agreement with the exact solution.

We solve this example with various values of h and k. Table 1 compares the L_∞-errors and temporal convergence orders with methods introduced in [2] for various values of k and $h=1/5$ on $[0,1]$. Table 2 compares the L_∞-errors and condition numbers with methods presented in [2] for various values of h and $k=1/100$ on $[0,1]$. In view of Table 2, we can observe that the proposed method is more well-conditioned than the methods in [2]. Figure 2 displays the approximate solution and related absolute errors with $h=1/5$ and $k=1/100$ on $[-6,6]$ using RBF-QR-FD method.

Table 1. The L_∞-errors and temporal convergence orders with $h = 1/5$, $\epsilon=0.4$ on $[0,1]\times[0,1]$ at $T=1$.

k	MLS		RBFK		RBFPS		RBF-QR-FD	
	L_∞	C-order	L_∞	C-order	L_∞	C-order	L_∞	C-order
1/10	$1.7749E-02$	–	$1.8527E-05$	–	$1.8527E-02$	–	$3.7588E-03$	–
1/20	$1.2346E-02$	0.5237	$1.2221E-02$	0.6003	$1.2221E-02$	0.6003	$7.7368E-04$	2.2805
1/40	$1.0716E-02$	0.2043	$1.1512E-02$	0.8622	$1.1512E-02$	0.8622	$2.2157E-04$	1.8040
1/80	$6.7502E-03$	0.6668	$7.6602E-03$	0.5877	$7.6602E-03$	0.5877	$4.5700E-05$	2.2775
1/160	$3.6208E-03$	0.8986	$4.2924E-03$	0.8356	$4.2924E-03$	0.8356	$1.5232E-05$	1.5851
1/320	$1.7386E-03$	1.0584	$2.1401E-03$	1.0041	$2.1401E-03$	1.0041	$1.1999E-05$	0.3442
1/640	$7.5607E-04$	1.0544	$2.9170E-04$	1.6748	$2.9170E-04$	1.6748	$1.1504E-05$	0.0608

Table 2. The L_∞-errors and condition numbers with $k = 1/100$, $\epsilon = 0.4$ and $n_s = 29$ on $[0,1] \times [0,1]$ at $T = 1$.

h	MLS		RBFK		RBFPS		RBF-QR-FD	
	L_∞	Cond (A)	L_∞	Cond(A)	L_∞	Cond(A)	L_∞	Cond(A)
1/5	$5.6015E-03$	$7.71E+04$	$3.4571E-03$	$3.45E+2$	$3.4571E-03$	1.25	$2.3804E-05$	1.0045
1/10	$8.4393E-03$	$6.02E+07$	$6.4590E-04$	$1.75E+3$	$6.4590E-04$	1.57	$3.6668E-05$	1.0614
1/15	$9.1597E-03$	$1.66E+10$	$2.9539E-04$	$2.95E+4$	$2.9539E-04$	2.10	$5.2778E-05$	1.0408
1/20	$9.3619E-03$	$3.02E+11$	$1.5029E-05$	$1.50E+4$	$1.5029E-05$	3.50	$3.4312E-05$	1.0760
1/25	$9.3872E-03$	$2.08E+12$	$8.9040E-05$	$8.90E+5$	$8.9040E-05$	4.97	$4.0460E-05$	1.1155

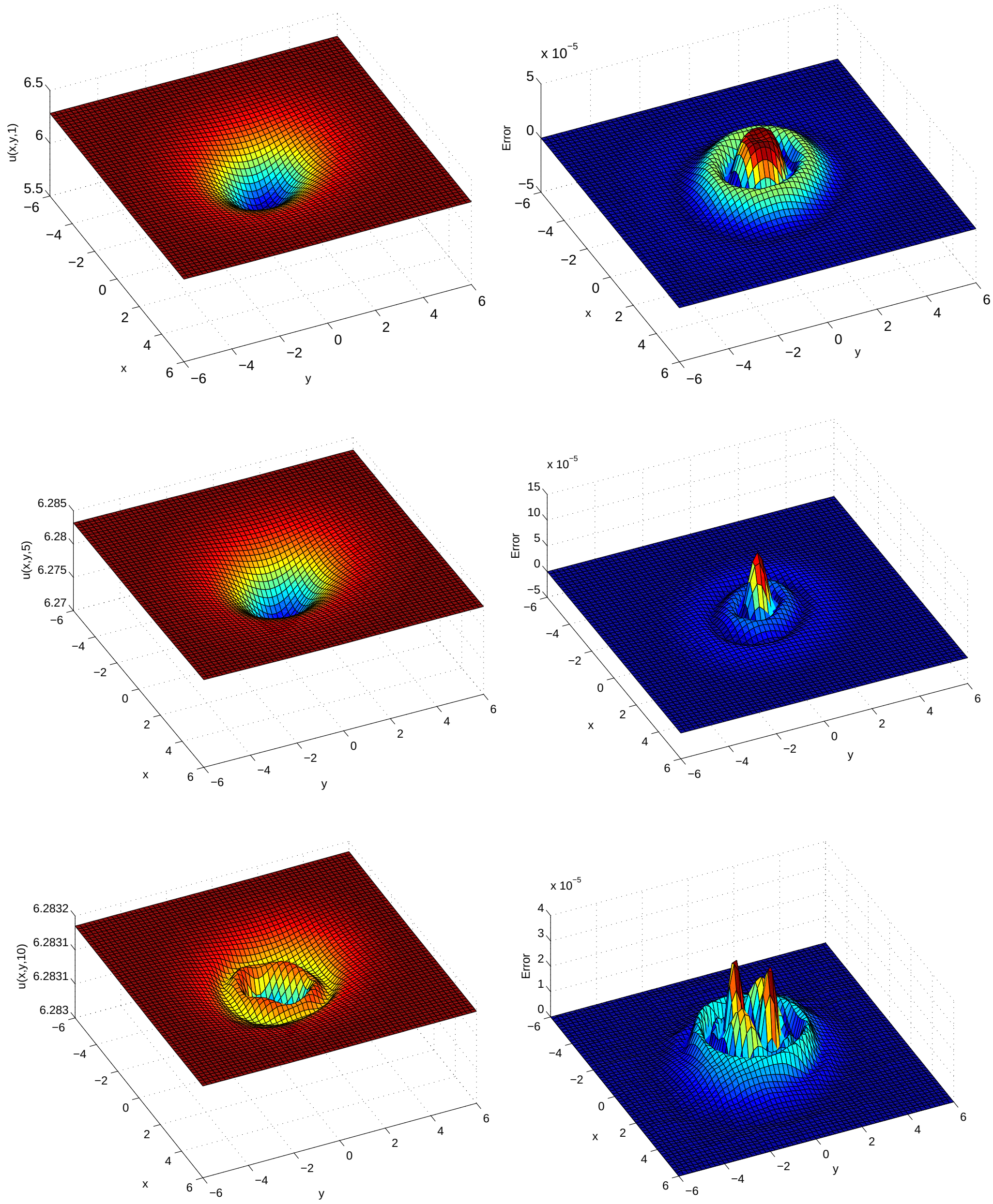

Figure 2. Surfaces of numerical solutions (left) and errors (right) with the RBF-QR-FD method when $\epsilon = 0.4$, $h = 1/5$, and $k = 1/100$ on $[-6,-6] \times [-6,6]$

References

[1] J. Rashidinia, MN. Rasoulizadeh, Radial basis function generated finite difference method for the solution of sinh-Gordon equation, TWMS Journal of Applied and Engineering Mathematics, 11(3) 893-905.

[2] M. Dehghan, M. Abbaszadeh, A. Mohebbi, The numerical solution of the two–dimensional sinh-Gordon equation via three meshless methods, Engineering Analysis with Boundary Elements 51 (2015) 220–235.

[3] G. Shechter, Matlab package kdtree.