On the stability of additive functional equations in probabilistic modular space

Shahin Yadegari[1,*], Mehdi Choubin[2]

[1]Master student of Mathematics, Velayat University, Iranshahr, Iran; yadegarishahin3@gmail.com
[2]Department of Mathematics, Velayat University, Iranshahr, Iran; m.choubin@velayat.ac.ir

Abstract. In this paper, we present a fixed point method to prove generalized Hyers–Ulam stability of the additive functional equation $f(x+y) = f(x)+f(y)$ in β-homogeneous probabilistic modular space.

Keywords: stability, functional equation, fixed point, probabilistic modular space.

AMS Mathematics Subject Classification [2010]: 39B52, 39B72, 47H09.

1. Introduction

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. Recall that the problem of stability of functional equations was motivated by a question of Ulam being asked in 1940 [8] and Hyers answer to it was published in [3]. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [7] for linear mappings by considering an unbounded Cauchy difference.

In this paper, we investigate the generalized Hyers–Ulame stability of additive functional equation for mappings from linear spaces into probabilistic modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [5]. In [2], after introducing the probabilistic modular, authors then investigated some basic facts in such spaces and study linear operators defined between them.

Definition 1.1. Let $\mathcal{X}$ be an arbitrary vector space.
(a) A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a modular if for arbitrary $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

*Speaker

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\rho$ given by $\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$. Let ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by $\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}$. A function modular is said to satisfy the Δ_2–condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \mathcal{X}_\rho$.

Definition 1.2. Let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then
(i) the sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of $\mathcal{X}_\rho$ is called ρ–complete complete if and only if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ–convergent to x.

Remark 1.3. Note that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.

We follow the definition of probabilistic modular space briefly as given in [2]. In the following, Δ stands for the set of all non-decreasing functions $f : \mathbb{R} \to \mathbb{R}_0^+$ satisfying $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$. We also denote the function min by $\wedge$.

Definition 1.4. A pair (X, μ) is called a probabilistic modular space ($\mathcal{PM}$-space) if X is a real vector space, μ is a mapping from X into Δ satisfying the following conditions:

(1) $\mu(x)(0) = 0$;
(2) $\mu(x)(t) = 1$ for all $t > 0$, if and only if $x = \theta \quad (\theta$ is the null vector in $X)$;
(3) $\mu(-x)(t) = \mu(x)(t)$;
(4) $\mu(\alpha x + \beta y)(s + t) \geq \mu(x)(s) \wedge \mu(y)(t)$, for all $x, y \in X$, and $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

For example, suppose that X is a real vector space and ρ is a modular on X. Define

$$\mu(x)(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\rho(x)}, & t > 0. \end{cases}$$

Then (X, μ) is a probabilistic modular space.

We say (X, μ) is β-homogeneous, where $\beta \in (0, 1]$ if,

$$\mu(\alpha x)(t) = \mu(x)\left(\frac{t}{|\alpha|^\beta}\right)$$

for every $x \in X$, $t > 0$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

Definition 1.5. Let (X, μ) be a $\mathcal{PM}$-space, $\{x_n\}$ be a sequence in X and $x \in X$. Then
(i) the sequence $\{x_n\}$, with $x_n \in (X, \mu)$, is μ–convergent to x and write $x_n \xrightarrow{\mu} x$, if for every $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer n_0 such that $\mu(x_n - x)(t) > 1 - \lambda$ for all $n \geq n_0$.
(ii) the sequence $\{x_n\}$, with $x_n \in (X, \mu)$, is μ–Cauchy, if for every $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer n_0 such that $\mu(x_n - x_m)(t) > 1 - \lambda$ for all $m, n \geq n_0$.

By [2], every μ-convergent sequence in a $\mathcal{PM}$-space is a μ-Cauchy sequence. If each μ-Cauchy sequence is μ-convergent in a $\mathcal{PM}$-space (X,μ), then (X,μ) is called a *μ-complete* $\mathcal{PM}$-space.
A $\mathcal{PM}$-space (X,μ) possesses Fatou property if for any sequence $\{x_n\}$ of X μ-converging to x, we have

$$\mu(x)(t) \geq \limsup_{n\geq 1} \mu(x_n)(t)$$

for each $t>0$.

Remark 1.6. Note that for any $x\in X$, $\mu(x)(.)$ is an increasing function, Since $\mu(x)\in\Delta$. Moreover, if μ is a β-homogeneous probabilistic modular on X and $x,y\in X$, then property (4) of Definition 1.4 shows that

$$\mu(x+y)\left(2^\beta(s+t)\right) = \mu\left(\frac{1}{2}x+\frac{1}{2}y\right)(s+t) \geq \mu(x)(s)\wedge\mu(y)(t).$$

For more details about the $\mathcal{PM}$-space, the readers refer to [6].

Our aim is based on the fixed point approach:

Theorem 1.7 ([4]). *Let X_ρ be a modular space such that satisfies the Fatou property. Let $\mathcal{C}$ be a ρ-complete nonempty subset of X_ρ and let $T:\mathcal{C}\to\mathcal{C}$ be quasicontraction, that is, there exists $K<1$ such that*

$$\rho(T(x)-T(y)) \leq K\max\{\rho(x-y),\rho(x-T(x)),\rho(y-T(y)),\rho(x-T(y)),\rho(y-T(x))\}.$$

Let $x\in\mathcal{C}$ such that $\delta_\rho(x):=\sup\{\rho(T^n(x)-T^m(x)) : m,n\in\mathbb{N}\}<\infty$. Then $T^n(x)$ ρ-converges to $\omega\in\mathcal{C}$. Moreover, if $\rho(\omega-T(\omega))<\infty$ and $\rho(x-T(\omega))<\infty$, then, the ρ-limit of $T^n(x)$ is a fixed point of T. Furthermore, if ω^* is any fixed point of T in $\mathcal{C}$ such that $\rho(\omega-\omega*)<\infty$, then one has $\omega=\omega*$.

Throughout this paper, we assume that μ is a probabilistic modular on X with the Fatou property(in the probabilistic modular sense) and (X,μ) is a μ-complete β-homogeneous $\mathcal{PM}$-space with $\beta\in(0,1]$.

2. Main Result

In this section, we establish the conditional stability of equation $f(x+y)=f(x)+f(y)$ in the $\mathcal{PM}$-spaces.

Theorem 2.1. *Let E be a linear space and (X,μ) be a μ-complete β-homogeneous $\mathcal{PM}$-space. Suppose $f:E\to(X,\mu)$ satisfies the condition $f(0)=0$ and an inequality of the form*

$$\mu\left(f(x+y)-f(x)-f(y)\right)(t) \geq \phi(x,y)(t) \tag{1}$$

for all $x,y\in E$, where $\phi:E\times E\to\Delta$ is a given function such that

$$\phi(2x,2x)(2^\beta Lt) \geq \phi(x,x)(t)$$

for all $x\in E$ and has the property

$$\lim_{n\to\infty}\phi(2^n x,2^n y)(2^{\beta n}t) = 1 \tag{2}$$

for all $x,y\in E$ and a constant $0<L<\frac{1}{2^\beta}$. Then there exists a unique additive mapping $j:E\to(X,\mu)$ such that

$$\mu(j(x)-f(x))\big(\frac{1}{1-2^\beta L}t\big) \geq \phi(x,x)(t) \tag{3}$$

for all $x \in E$.

Proof. We consider the set $\mathcal{M} = \{h : E \to (X, \mu) | \quad h(0) = 0\}$ and introduce the modular ρ on $\mathcal{M}$ as follows,

$$\rho(h) = \inf\{c > 0 : \mu(h(x))(ct) \geq \phi(x, x)(t)\}.$$

It is clear that ρ is even and $\rho(0) = 0$. If $\rho(h) = 0$, then for each $c > 0$, $\mu(h(x))(ct) \geq \phi(x, x)(t)$ for all $t > 1$, $x, y \in E$. Now if $\epsilon = ct$ is fixed, and $t \to +\infty$, then $\mu(h(x))(\epsilon) = 1$, which implies that $h = 0$. It is sufficient to show that ρ satisfies the following condition $\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)$, if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. Let $\varepsilon > 0$ be given. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \rho(g) + \varepsilon; \quad \mu(g(x))(c_1 t) \geq \phi(x, x)(t)$$

and

$$c_2 \leq \rho(h) + \varepsilon; \quad \mu(h(x))(c_2 t) \geq \phi(x, x)(t).$$

If $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, then we get

$$\mu(\alpha g(x) + \beta h(x))(c_1 t + c_2 t) \geq \mu(g(x))(c_1 t) \wedge \mu(h(x))(c_2 t) \geq \phi(x, x)(t),$$

whence $\rho(\alpha g + \beta h) \leq c_1 + c_2 \leq \rho(g) + \rho(h) + 2\varepsilon$. Hence, we have $\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)$. We now show that ρ satisfies the Δ_2–condition with $\kappa = 2^\beta$. For $\varepsilon > 0$ given, there exists $c > 0$ such that $c \leq \rho(h) + \varepsilon$ and $\mu(h(x))(ct) \leq \phi(x, x)$. Since (X, μ) is a β-homogeneous $\mathcal{PM}$-space., we have $\mu(2h(x))(2^\beta ct) = \mu(h(x))(ct) \geq \phi(x, x)$, whence $\rho(2h) \leq 2^\beta c \leq 2^\beta \rho(h) + 2^\beta \varepsilon$. Therefore $\rho(2h) \leq 2^\beta \rho(h)$. Thus ρ satisfies the Δ_2–condition with $\kappa = 2^\beta$. Moreover, ρ satisfies the Fatou property(in the modular sense). Indeed, if sequence $\{h_n\}$ of $\mathcal{M}$ ρ-converging to h, then we can easy see that $h(x)$ μ-converging to $h(x)$ for any $x \in E$. Let $\varrho := \liminf \rho(h_n) < \infty$ and $\rho(h) > \varrho$. We have $\mu(h(x))(\varrho t) < \phi(x, x)(t)$ for all $t > 0$. Since μ satisfies the Fatou property(in the probabilistic modular sense), hence

$$\limsup \mu(h_n(x))(\varrho t) \leq \mu(h(x))(\varrho t) < \phi(x, x)(t).$$

By last inequality we get there exists a positive integer $n_0 \in \mathbb{N}$ such that $\mu(h_n(x))(\varrho t) < \phi(x, x)(t)$ and so $\rho(h_n) > \varrho$ for all $n \geq n_0$. Thus $\liminf \rho(h_n) > \varrho$, this is a contradiction. Therefore, ρ satisfies the Fatou property.

If $\delta > 0$ and $\lambda \in (0, 1)$ are given, since $\phi(x, x) \in \Delta$, there exists $t_0 > 0$ such that $\phi(x, x)(t_0) > 1 - \lambda$. Let $\{h_n\}$ be a ρ–Cauchy sequence in $\mathcal{M}_\rho$ and let $\varepsilon < \frac{\delta}{t_0}$ be given. There exists a positive integer $n_0 \in \mathbb{N}$ such that $\rho(h_n - h_m) \leq \varepsilon$ for all $n, m \geq n_0$. Now by considering the definition of the modular ρ, we see that

$$\mu(h_n(x) - h_m(x))(\delta) \geq \mu(h_n(x) - h_m(x))(\varepsilon t_0) \geq \phi(x, x)(t_0) > 1 - \lambda, \tag{4}$$

for all $x \in E$ and $n, m \geq n_0$. If x is a arbitrary given points of E, (4) implies that $\{h_n(x)\}$ is a μ–Cauchy sequence in (X, μ). Since (X, μ) is μ–complete, so $\{h_n(x)\}$ is μ–convergent in (X, μ), for all $x \in E$. Hence, we can define a function $h : E \to (X, \mu)$ by $h(x) = \lim_{n\to\infty} h_n(x)$, for any $x \in E$. Let m increase to infinity, then (4) implies that $\rho(h_n - h) \leq \varepsilon$ for all $n \geq n_0$, since μ has the Fatou property. Thus $\{h_n\}$ is ρ–convergent sequence in $\mathcal{M}_\rho$. Therefore $\mathcal{M}_\rho$ is ρ–complete. Now, letting $x = y$ in (1), we get

$$\mu(f(2x) - 2f(x))(t) \geq \phi(x, x)(t) \tag{5}$$

and so

$$\mu\left(\frac{f(2x)}{2} - f(x)\right)\left(\frac{t}{2^\beta}\right) \geq \phi(x, x)(t) \tag{6}$$

for all $x \in E$. If we replace x by $2x$ in (5) we get

$$\mu(f(4x) - 2f(2x)) \geq \phi(2x, 2x)(t) \tag{7}$$

for all $x \in \mathcal{E}$. Since μ is a β-homogeneous probabilistic modular, by (7) we obtain

$$\begin{aligned} \mu\left(\frac{f(4x)}{2} - f(2x)\right)(Lt) &= \mu\left(\frac{1}{2}\{f(4x) - 2f(2x)\}\right)(Lt) \\ &= \mu(f(4x) - 2f(2x))(2^\beta Lt) \geq \phi(2x, 2x)(2^\beta Lt) \geq \phi(x, x)(t), \end{aligned} \tag{8}$$

for all $x \in E$. Therefore, by (5) and (8) we get

(9)
$$\begin{aligned} &\mu\left(\frac{f(2^2x)}{2^2} - f(x)\right)(Lt + t) = \mu\left(\frac{1}{2}\left\{\frac{f(4x)}{2} - f(2x)\right\} + \frac{1}{2}\{f(2x) - 2f(x)\}\right)(Lt + t) \\ &\geq \mu\left(\frac{f(4x)}{2} - f(2x)\right)(Lt) \wedge \mu\left(f(2x) - 2f(x)\right)(t) \geq \phi(x, x)(t), \end{aligned}$$

for all $x \in E$. By replacing x and $Lt + t$ by $2x$ and $2^\beta(L^2t + Lt)$ in (9) , respectively, we get

$$\begin{aligned} \mu\left(\frac{f(2^3x)}{2^3} - \frac{f(2x)}{2}\right)(L^2t + Lt) &= \mu\left(\frac{f(2^3x)}{2^2} - f(2x)\right)\left(2^\beta(L^2t + Lt)\right) \\ &\geq \phi(2x, 2x)(2^\beta Lt) \geq \phi(x, x)(t), \end{aligned} \tag{10}$$

and moreover, by (6) and (10) we get

$$\begin{aligned} &\mu\left(\frac{f(2^3x)}{2^3} - f(x)\right)\left(2^\beta(L^2t + Lt) + t\right) \\ &\geq \mu\left(\frac{f(2^3x)}{2^3} - \frac{f(2x)}{2}\right)(L^2t + Lt) \wedge \mu\left(\frac{f(2x)}{2} - f(x)\right)\left(\frac{t}{2^\beta}\right) \geq \phi(x, x)(t), \end{aligned}$$

for all $x \in E$. By mathematical induction, we can easily see that

$$\mu\left(\frac{f(2^nx)}{2^n} - f(x)\right)\left(\left\{2^{\beta(n-2)}L^{n-1} + \sum_{i=1}^{n-1}(2^\beta L)^{i-1}\right\} t\right) \geq \phi(x, x)(t), \tag{11}$$

for all $x \in E$ and $n \geq 3$. Next, we consider the function $\mathcal{T} : \mathcal{M}_\rho \to \mathcal{M}_\rho$ defined by $\mathcal{T}h(x) := \frac{1}{2}h(2x)$, for all $h \in \mathcal{M}_\rho$. Let $g, h \in \mathcal{M}_\rho$ and let $c \in [0, \infty]$ be an arbitrary constant with $\rho(g-h) \leq c$. From the definition of ρ, we have $\mu(g(x)-h(x))(ct) \geq \phi(x, x)(t)$ for all $x \in E$. By the assumption and the last inequality, we get

$$\mu\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right)(Lct) = \mu(g(2x) - h(2x))(2^\beta Lct) \geq \phi(2x, 2x)(2^\beta Lt) \geq \phi(x, x)(t)$$

for all $x \in E$. Hence, $\rho(\mathcal{T}g - \mathcal{T}h) \leq L\rho(g - h)$, for all $g, h \in \mathcal{M}_\rho$ that is, $\mathcal{T}$ is a ρ–strict contraction. We show that the ρ–strict mapping $\mathcal{T}$ satisfies the conditions of Theorem 1.7.

Next, we assert that $\delta_\rho(f) = \sup\left\{\rho\left(\mathcal{T}^n(f) - \mathcal{T}^m(f)\right); n, m \in \mathbb{N})\right\} < \infty$. By (11) we get

$$\rho\left(\mathcal{T}^n(f) - f\right) \leq 2^{\beta(n-2)}L^{n-1} + \sum_{i=1}^{n-1}(2^\beta L)^{i-1} \leq \sum_{i=1}^{n}(2^\beta L)^{i-1} \leq \frac{1}{1 - 2^\beta L} \tag{12}$$

Since ρ satisfies the Δ_2–condition with $\kappa = 2^\beta$, it follows from inequality (12) that

$$\rho\left(\mathcal{T}^n(f) - \mathcal{T}^m(f)\right) \leq \frac{1}{2}\rho\left(2\mathcal{T}^n(f) - 2f\right) + \frac{1}{2}\rho\left(2\mathcal{T}^m(f) - 2f\right)$$
$$\leq \frac{\kappa}{2}\rho\left(\mathcal{T}^n(f) - f\right) + \frac{\kappa}{2}\rho\left(\mathcal{T}^m(f) - f\right) \leq \frac{2^\beta}{1 - 2^\beta L}\phi(x, x)(t),$$

for every $x \in E$ and $n, m \in \mathbb{N}$, which implies that $\rho\left(\mathcal{T}^n(f) - \mathcal{T}^m(f)\right) \leq \frac{2^\beta}{1-2^\beta L}$, for all $n, m \in \mathbb{N}$. By the definition of $\delta_\rho(f)$, we have $\delta_\rho(f) < \infty$. Theorem 1.7 shows that $\{\mathcal{T}^n(f)\}$ is ρ–converges to $j \in \mathcal{M}_\rho$. Since ρ has the Fatou property inequality (12), gives $\rho(\mathcal{T}j - f) < \infty$.

If we replace m by $n+1$ in inequality (**??**), then we obtain $\rho\left(\mathcal{T}^{n+1}f - \mathcal{T}^n f\right) \leq \frac{2^\beta}{1-2^\beta L}$, for all $x \in E$. Therefore $\rho(\mathcal{T}(j) - j) \leq (2^\beta/1 - 2^\beta L) < \infty$. It follows from Theorem 1.7 that ρ–limit of $\{\mathcal{T}^n(f)\}$ i.e., $j \in \mathcal{M}_\rho$ is fixed point of map $\mathcal{T}$. If we replace x by $2^n x$ and y by $2^n y$ in inequality (1), then we obtain

$$\mu\left(f(2^n(x+y)) - f(2^n x) - f(2^n y)\right)(t) \geq \phi(2^n x, 2^n y)(t)$$

for all $x, y \in E$. Hence,

$$\mu\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right)(t) \geq \rho\left(f(2^n(x+y)) - f(2^n x) - f(2^n y)\right)(2^{\beta n} t)$$
$$\geq \phi(2^n x, 2^n y)(2^{\beta n} t)$$

for all $x, y \in E$. Taking the limit, we deduce that $j(x+y) = j(x) + j(y)$ for all $x, y \in E$. It follows from inequality (12) that $\rho(j - f) \leq \frac{1}{1-2^\beta L}$. If j^* is another fixed point of $\mathcal{T}$, then

$$\rho(j - j^*) \leq \frac{1}{2}\rho(2\mathcal{T}(j) - 2f) + \frac{1}{2}\rho(2\mathcal{T}(j^*) - 2f)$$
$$\leq \frac{\kappa}{2}\rho(\mathcal{T}(j) - f) + \frac{\kappa}{2}\rho(\mathcal{T}(j^*) - f) \leq \frac{2^\beta}{1 - 2^\beta L} < \infty.$$

Since $\mathcal{T}$ is ρ–strict contraction, we get $\rho(j - j^*) = \rho(\mathcal{T}(j) - \mathcal{T}(j^*)) \leq L\rho(j - j^*)$, which implies that $\rho(j - j^*) = 0$ or $j = j^*$, since $\rho(j - j^*) < \infty$. This prove the uniqueness of j. □

References

1. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan 2 (1950), 64–66.
2. K. Fallahi, K. Nourouzi, *Probabilistic modular spaces and Linear operators*, Acta Appl Math. 105, (2009) 123-140.
3. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222–224.
4. M. A. Khamsi, *Quasicontraction Mapping in modular spaces without Δ_2–condition*, Fixed Point Theory and Applications Volume (2008), Artical ID 916187, 6 pages.
5. H. Nakano, *Modulared Semi-Ordered Linear Spaces*, in: Tokyo Math. Book Ser., Vol. **1**, Maruzen Co., Tokyo, 1950.
6. K. Nourouzi, *Probabilistic modular spaces*, Further progress in analysis, World Sci. Publ., Hackensack, (2009) 814-818.
7. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297–300.
8. S. M. Ulam, *Problems in Modern Mathematics*, Chapter IV, Science Editions, Wiley, New York, 1960.