A generalized Chen distribution with application in reliability

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Abstract. In this paper, a new flexible family of distribution is proposed for considering as a class of lifetime distribution. After introducing the general class, an special model of the new family, namely, Arcsine Exponentiated Odd Chen Fréchet (AEOCh-Fr) distribution is discussed. Some statistical and reliability properties of the distribution are investigated. The model parameters are estimated by the maximum likelihood method. Finally, a real data application about the fatigue life is used to illustrate the usefulness of the proposed distribution in modeling real data.

Keywords: Chen distribution, Fréchet distribution, hazard rate function, maximum likelihood estimation.

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1. Introduction

In statistical literature, researchers have made many efforts to define new families or extend existing distributions and introduce flexible classes to model data in several areas such as engineering, medical sciences, environmental, biological studies, life testing problems, demography, actuarial and economics. Some researcher have proposed some classes by adding one or more parameters to generate new distributions. However, in many practical situations, classical distributions do not provide adequate fits to real data. So, there is a clear need for extended forms of these distributions in many applied areas such as survival analysis, reliability theory, insurance, medical sciences. In the last decades, interest in developing more flexible distributions has been increased, for example see [1, 4–6, 8–10] and references therein.

Chen [2] proposed a new two-parameter lifetime distribution with cumulative distribution function (cdf) and probability density function (pdf)

$$W(x;\alpha,\beta) = 1 - \mathrm{e}^{-\alpha\left[\mathrm{e}^{x^{\beta}}-1\right]}, \quad x > 0,$$

and

$$w(t;\alpha,\beta) = \alpha\beta x^{\beta-1}\exp\left(x^{\beta}\right)\exp\left(-\alpha\left[\exp\left(x^{\beta}\right)-1\right]\right), \quad x > 0,$$

respectively, where $\alpha > 0$ and $\beta > 0$. Chaubey and Zhang [3] introduced a flexible model called exponentiated Chen (ECh) distribution. The cdf and pdf of ECh distribution are given by

$$K(x;\alpha,\beta,\theta)=\left(1-\mathrm{e}^{-\alpha\left[\mathrm{e}^{x^{\beta}}-1\right]}\right)^{\theta},\quad x>0,$$

and

$$k(x;\alpha,\beta,\theta)=\alpha\theta\beta x^{\beta-1}\mathrm{e}^{x^{\beta}}\exp\left(-\alpha\left[\mathrm{e}^{x^{\beta}}-1\right]\right)\left(1-\mathrm{e}^{-\alpha\left[\mathrm{e}^{x^{\beta}}-1\right]}\right)^{\theta-1},\quad x>0,$$

respectively, where $\alpha,\beta,\theta>0$ are shape parameters. Eliwa et al. [6] proposed a new flexible class of distributions called exponentiated odd Chen-G (EOCh-G) family. The cdf of EOCh-G family of distributions is defined by

$$(1)\qquad F(x;\alpha,\beta,\theta,\boldsymbol{\kappa})=\left[1-\exp\left(-\alpha\left[e^{\left(\frac{G(x;\boldsymbol{\kappa})}{1-G(x,\boldsymbol{\kappa})}\right)^{\beta}}-1\right]\right)\right]^{\theta},\quad x\in\mathbb{R},$$

where $\alpha,\beta,\theta>0$ and $G(x;\boldsymbol{\kappa})$ represents the cdf of a baseline model with parameter vector $\boldsymbol{\kappa}$. The associated pdf is

$$\begin{aligned}f(x;\alpha,\beta,\theta,\boldsymbol{\kappa})=&\frac{\alpha\theta\beta g(x;\boldsymbol{\kappa})G(x;\boldsymbol{\kappa})^{\beta-1}}{(1-G(x;\kappa))^{\beta+1}}\mathrm{e}^{\left(\frac{G(x;\boldsymbol{\kappa})}{1-G(x;\boldsymbol{\kappa})}\right)^{\beta}}\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{G(x;\boldsymbol{\kappa})}{1-G(x;\boldsymbol{\kappa})}\right)^{\beta}}-1\right]\right)\\&\times\left[1-\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{G(x;\boldsymbol{\kappa})}{1-G(x;\boldsymbol{\kappa})}\right)^{\beta}}-1\right]\right)\right]^{\theta-1},\quad x\in\mathbb{R},\end{aligned}$$

where $g(x;\boldsymbol{\kappa})$ represents the pdf of a baseline model. In this paper, we introduce a new family of distributions called the arcsine exponentiated odd Chen-G (AEOCh-G) distribution by applying the approach of arcsine-X family of distributions (Tung et al. [10]) to the EOCh-G family. The cdf of the arcsine-X distributions is given by

$$(2)\qquad G^{*}(x;\boldsymbol{\kappa})=\frac{2}{\pi}\arcsin\left[G(x;\boldsymbol{\kappa})\right],\quad x\in\mathbb{R}.$$

The rest of the paper is organized as follows. In Section 2, some statistical properties of the AEOCh-G distributions such as failure rate and moments are presented. Estimation of the parameters of AEOCh-G distribution, in an special case when baseline distribution is Fréchet, by the method of maximum likelihood are studied in Section 3. In Section 4, a real data application of fatigue life data are illustrated the potential of AEOCh-G distributions compared with other distributions.

2. Statistical Properties of the distribution

The cdf of the proposed AEOCh-G distributions is obtained by using the expression 2 in 1 as

$$(3)\qquad F(x;\alpha,\beta,\theta,\boldsymbol{\kappa})=\left[1-\exp\left(-\alpha\left[e^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]\right)\right]^{\theta},\quad x\in\mathbb{R}.$$

The probability density function (pdf) of the AEOCh-G distributions takes the form

$$\begin{aligned}(4)\qquad f(x;\alpha,\beta,\theta,\boldsymbol{\kappa})=&\frac{\pi\alpha\theta\beta g(x;\boldsymbol{\kappa})}{2\sqrt{(1-G^{2}(x;\kappa))}\arcsin^{2}\left[G(x,\boldsymbol{\kappa})\right]}\left(\frac{\pi}{2\arcsin\left[G(x,\boldsymbol{\kappa})\right]}-1\right)^{-\beta-1}\\&\times\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]\right)\end{aligned}$$

$$\times \left[1-\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]\right)\right]^{\theta-1}.$$

Failure rate function plays a key role in applied probability and in the theory of reliability. The failure rate function of a random variable X is defined as

$$h(x)=\frac{f(x)}{\bar{F}(x)}.$$

Considering the random variable X as a lifetime random variable, the failure rate $h(x)$ represents the likelihood that X be realized right after time x, given that it was not realized up to time x. The failure rate function corresponding to (4) is

$$\begin{aligned} h(x;\alpha,\beta,\theta,\boldsymbol{\kappa}) =& \frac{\pi\alpha\theta\beta g(x;\boldsymbol{\kappa})}{2\sqrt{(1-G^2(x;\kappa))}\arcsin^2[G(x,\boldsymbol{\kappa})]}\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta-1}\\ &\times\left[1-\left[1-\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]\right)\right]^{\theta}\right]^{-1}\\ &\times\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]+\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}\right)\\ &\times\left[1-\exp\left(-\alpha\left[\mathrm{e}^{\left(\frac{\pi}{2\arcsin[G(x,\boldsymbol{\kappa})]}-1\right)^{-\beta}}-1\right]\right)\right]^{\theta-1}. \end{aligned}$$

The inverse of the cdf of the AEOCh-G family yields the quantile function

$$Q(u)=G^{-1}\left[\sin\left(\frac{\pi/2}{1+\left\{\log\left(1-\frac{1}{\alpha}\log\left[1-u^{\frac{1}{\theta}}\right]\right)\right\}^{-1/\beta}}\right)\right],\ u\in(0,1), \tag{5}$$

where G^{-1} represents the baseline quantile function. The above function facilitates ready quantile-based statistical modeling [?]. In addition, $Q(u)$ gives a trivial random variable generation: if $U\sim\mathcal{U}(0,1)$, then

$$Q(u)=G^{-1}\left[\sin\left(\frac{\pi/2}{1+\left\{\log\left(1-\frac{1}{\alpha}\log\left[1-X^{\frac{1}{\theta}}\right]\right)\right\}^{-1/\beta}}\right)\right],\ u\in(0,1),$$

follows AEOCh-G distribution. The median of the distribution can be derived from (5) by setting $u=0.5$.

3. An special AEOCh-G model

Let the baseline distribution in (3) be the Fréchet (Fr) distribution with parameters a and b. Then, the cdf of the AEOCh-Fr distribution is given by

$$F(x;\alpha,\beta,\theta,a,b)=\left[1-\exp\left(-\alpha\left[e^{\left(\frac{\pi}{2\arcsin[\exp-(b/x)^{a}]}-1\right)^{-\beta}}-1\right]\right)\right]^{\theta},\quad x>0,$$

where the parameters α,β,θ and a control the shapes of the distribution and b is the scale parameter. If a random variable X has the AEOCh-Fr distribution, then we write $X\sim$ AEOCh-Fr$(\alpha,\beta,\theta,a,b)$. Figures 1 and 2 shows the pdfs and failure rate functions of the AEOCh-Fr for various values of the parameters, respectively. Furthermore, the failure rate function can be either decreasing, increasing, unimodal-bathtub or of bathtub shape, which makes the distribution more flexible to fit different lifetime data sets.

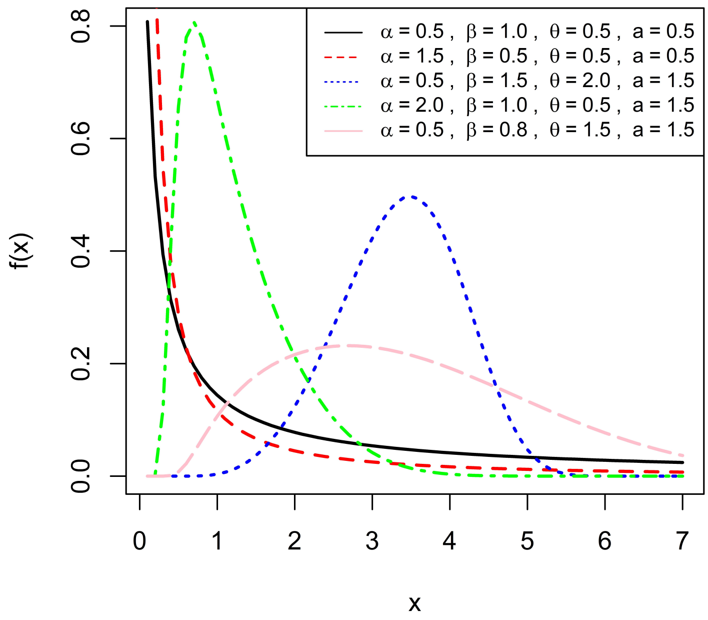

FIGURE 1. Probability density functions of AEOCh-Fr distribution for $b = 1.5$

4. Maximum likelihood estimation

In this section, we consider the estimation of the parameters of the AEOCh-Fr distribution by the method of maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample of size n of the RT distribution with unknown parameter vector $\Theta = (\alpha, \beta, \theta, a, b)^T$. The log-likelihood function for $\boldsymbol{\theta}$ based on a given random sample is

$$\begin{aligned}
\ell(\Theta) =& n\log(\frac{\pi}{2}\alpha\beta\theta b^a) + (\theta-1)\sum_{i=1}^{n}\log\left(1-\exp\left(-\alpha\left[e^{\left(\frac{\pi}{2\arcsin\left[\exp-(b/x_i)^a\right]}-1\right)^{-\beta}}-1\right]\right)\right) \\
&-\alpha\sum_{i=1}^{n}\exp\left[\left(\frac{\pi}{2\arcsin\left[\exp-(b/x_i)^a\right]}-1\right)^{-\beta}-1\right]+\sum_{i=1}^{n}\left(\frac{\pi}{2\arcsin\left[\exp-(b/x_i)^a\right]}-1\right)^{-\beta} \\
&-(\beta+1)\sum_{i=1}^{n}\log\left(\frac{\pi}{2\arcsin\left[\exp-(b/x_i)^a\right]}-1\right)-(a+1)\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left(\frac{b}{x_i}\right)^a \\
&-\frac{1}{2}\sum_{i=1}^{n}\log\left(1-\exp\left(\frac{b}{x_i}\right)^{2a}\right)-2\sum_{i=1}^{n}\log\left(\arcsin\left[\exp\left(\frac{b}{x_i}\right)^a\right]\right).
\end{aligned}$$

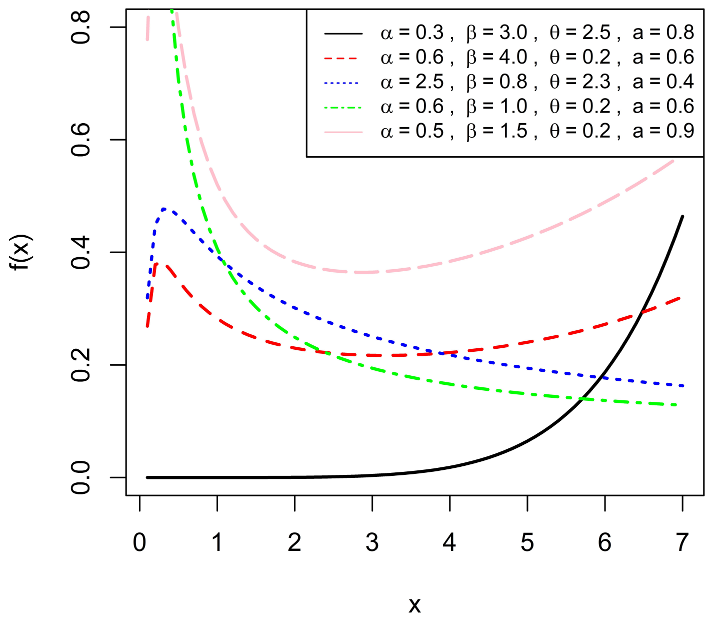

FIGURE 2. Failure rate functions of AEOCh-Fr distribution for $b = 1.5$

The maximum likelihood estimates (MLEs) of the unknown parameters are obtained by maximizing $\ell(\Theta)$ with respect to Θ. The MLE $\widehat{\Theta} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\theta}, \widehat{a}, \widehat{b})^T$ of $\Theta = (\alpha, \beta, \theta, a, b)^T$ can be obtained by solving the following equations simultaneously:

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \frac{\partial \ell(\Theta)}{\partial \beta} = \frac{\partial \ell(\Theta)}{\partial \theta} = \frac{\partial \ell(\Theta)}{\partial a} = \frac{\partial \ell(\Theta)}{\partial b} = 0.$$

There is no closed-form expression for the MLEs, so nonlinear optimization algorithms such as Newton-Raphson iterative technique can be applied to solve the equations and obtain the estimate $\widehat{\boldsymbol{\theta}}$ numerically.

5. Application

In the following, we present an application of the proposed AEOCh-Fr distribution to a lifetime data set. This application will show the flexibility of this distribution in modeling real data. The data set represents the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillates at 18 cycles per second (cps), see Shanker and Shukla [7].

we compare the fits of the fits of the EOChFr distribution with some competitive models like odd Chen Fr (OChFr), Type I generalized exponential Fr (TIGEFr), odd flexible Weibull Fr (OFWFr), Topp-Leaon Fr (ToLeFr), exponentiated Gompertz Fr (EGoFr), exponentiated transmuted Fr (ETrFr), transmuted Fr (TrFr), Gumbel Fr (GuFr), exponentiated Fr (EFr) and Fr. Table 1 gives the values of the following goodness of fit statistics for the considered models: the negative maximized log-likelihood $(-\ell(\widehat{\Theta}))$, Cramér–von

Mises (CvM), Anderson–Darling (AD) statistics, and Kolmogorov-Smirnov (KS) statistic and its p-value. The statistical packages are used by R 4.1.1 to obtain numerical results.

TABLE 1. Goodness-of-fit statistics for fatigue times.

Model	$-L$	CvM	AD	KS	palue
AEOCh-Fr	452.550	0.326	0.048	0.063	0.818
EOChFr	456.089	0.340	0.054	0.065	0.786
OChFr	456.320	0.356	0.056	0.068	0.732
TIGEFr	475.191	2.497	0.433	0.133	0.057
OFWFr	459.69	0.672	0.099	0.090	0.383
ToLeFr	466.35	1.543	0.992	0.121	0.102
EGoFr	461.298	0.992	0.177	0.107	0.198
TrFr	466.41	1.565	0.275	0.120	0.105
GuFr	475.73	2.558	0.443	0.135	0.050
EFr	475.18	2.497	0.433	0.133	0.056
Fr	475.18	2.497	0.433	0.133	0.056

From the values of these statistics, we infer that the AEOCh-Fr distributions provides a better fit than other distributions for the real data set.

6. Conclusion

In this paper, a new family of distributions called the AEOCh-G family is proposed. An special subset of this family is investigated by considering the Fréchet distribution. Finally, the distribution is fitted to a real data set to illustrate the applicability and usefulness of this distribution in modeling data.

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