On the Sum of K-Frames in Hilbert Spaces

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Abstract. In recent years, K-frame was presented to reconstruct elements from the range of a bounded linear operator K in a separable Hilbert space. K-frames are more generalization than the ordinary frames and many properties of ordinary frames may not holds for such generalization of frames. In this paper, we obtain some sufficient conditions for the finite sum of K-frames to be a K^2-frame.

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1. Introduction

In 1952, Duffin and Schaeffer introduced frames in Hilbert spaces in their fundamental paper [4]. Later in 1986, the formal definition of frame in the abstract Hilbert spaces were given by Daubechies, Grossman, Meyer [3]. However, a frame has its own characteristics. Unlike a basis, the expansion of element in space is not unique. It is precisely because of this feature of a frame that its construction is more flexible [11]. The frames have been applied in many fields such as signal and image processing, [2]. coding theory [10], data transmission technology [9, 11]. With the development of the frames theory, some special frames such as fusion frames [13], G-frames [1], K-frames [17] are put forward.

K-frames, were recently introduced by Gavruta in [6] to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. Now K-frames have been studied by many authors. Note that many properties for K-frames are quite different to the classical frames. For instance, in general the synthesis operators for K-frames are not surjective, the frame operators for K-frames in Hilbert space H are not invertible on H. Unlike a traditional frame, the spanning scope of a K-frame was limited to the range of operator K [7]. In reality, the K-frames expand the meanings of the traditional frames [17]. Therefore, K-frames have been studied by many scholars. For example, Xiao X. C. Li liang etc studied a relationship between K-frames and operator K, and used the synthesis operator and the operator K to characterize the optimal boundary of K-frames in [14]. In [12] and [15], the authors studied the sum and of G-frames in Hilbert spaces.

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And in [8, 16], the authors studied the sum of K-frames. They discussed the sum of a K-frame and a Bessel sequence and obtain some sufficient conditions for the sum of a K-frame and a Bessel sequence to be a K-frame. In this paper, we discuss the sum of K-frames for Hilbert spaces. Indeed, we obtain some sufficient conditions for the finite sum of K-frames to be a K^2-frame.

Throughout this paper, H will denote a separable Hilbert space with the inner product $< .,. >$ and $B(H)$ denote the space of all bounded linear operator on H. We also denote $\mathcal{R}(T)$ for range set of T where $T \in B(H)$ and ℓ^2 denote the space of square summable scalar-valued sequences with index set $\mathbb{N}$.

Definition 1.1. Let H be a separable Hilbert space. A collection of vectors $F = \{f_i\}_{i=1}^k \subset H$ is called a Bessel sequence for H if there exists a constant $0 < B$ such that

$$\sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for any $f \in H$.

When the Bessel sequence satisfies certain conditions, it can become a frame.

Definition 1.2. Let H be a separable Hilbert space. A collection of vectors $F = \{f_i\}_{i=1}^k \subset H$ is called a frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for any $f \in H$.

Definition 1.3. Let H be a separable Hilbert space and $K \in B(H)$. A collection of vectors $F = \{f_i\}_{i=1}^k \subset H$ is called a K-frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for any $f \in H$.

The constants A and B are called lower and upper frame bound for the K-frame. In particular, if $A\|K^*f\|^2 = \sum_{i=1}^{k} |\langle f, f_i\rangle|^2$, for any $f \in H$, then the K-frame $F = \{f_i\}_{i=1}^k \subset H$ is called a tight K-frame for H. If $A = 1$, then the tight K-frame $F = \{f_i\}_{i=1}^k \subset H$ is called a Parseval K-frame for H.
Sometimes a sequence is not necessarily a frame, but it can be a K-frame. In other words, the K-frames are more flexible than frames.

Definition 1.4. [12] A sequence $\{a_i\}_{i\in I}$ is said to be positively confined if

$$0 < \inf_{i\in I} a_i \leq \sup_{i\in I} a_i < +\infty.$$

Then let us introduce the following three operators which often appear through out all this paper:

Definition 1.5. Let $F = \{f_i\}_{i=1}^k \subset H$ be a Bessel sequence for H.

(I) The analysis operator of F is defined by

$$T : H \longrightarrow \ell^2, \qquad Tf = \{\langle f, f_i\rangle\}_{i=1}^k.$$

(II) The synthesis operator of F is defined by

$$T^* : \ell^2 \longrightarrow H, \qquad T\{c_i\}_{i=1}^k = \sum_{i=1}^k c_i f_i.$$

(III) The frame operator of F is defined by

$$S : H \longrightarrow H, \qquad Sf = T^*Tf = \sum_{i=1}^k \langle f, f_i\rangle f_i.$$

Sometimes the analysis operator of F is defined by

$$T : H \longrightarrow \ell^2, \qquad Tf = \sum_{i=1}^k \langle f, f_i\rangle e_i,$$

where the $\{e_i\}_{i=1}^k$ is the canonical basis of ℓ^2.

Theorem 1.6 ([5]). *Let $S, T, U \in B(H)$. Then the following are equivalent:*

(I) $\mathcal{R}(S) \subseteq \mathcal{R}(T) + \mathcal{R}(U)$.
(II) $SS^* \leq \lambda^2(TT^* + UU^*)$ *for some* $\lambda > 0$.
(III) $S = TA + UB$ *for some* $A, B \in B(H)$.

The following results provide some sufficient conditions for the finite sum of K-frames to be a K-frame.

Theorem 1.7 ([8]). *Let $\{f_i\}_{i\in I}$ be a K-frame for H with bounds A, B. K is a bounded linear operator in H. Let $\{g_i\}_{i\in I}$ be a Bessel sequence with analysis operator T_G. If there is a positive constant c that makes $\|T_G f\| \leq c\|K^* f\|$, then for any two positively confined sequences $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$, if $A \inf_{i\in I} a_i^2 > 2c^2 \sup_{i\in I} b_i^2$, then $\{a_i f_i + b_i g_i\}_{i\in I}$ is a K-frame for H.*

Corollary 1.8 ([8]). *Let $\{f_i\}_{i\in I}$ be a frame for H with bounds A, B. Let $\{g_i\}_{i\in I}$ be a Bessel sequence with analysis operator T_G. For any two positively confined sequences $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$, if $\|T_G\|^2 < \dfrac{A \inf_{i\in I} a_i^2}{2 \sup_{i\in I} b_i^2}$, then $\{a_i f_i + b_i g_i\}_{i\in I}$ is a frame for H.*

Theorem 1.9 ([8]). *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two K-frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively. Let T_F and T_G be analysis operators of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively. Let $U, V \in B(H)$ with $UK = KU$ and $\|U^* f\| \geq \|f\|$. If there is a positive constant c such that $\|K^* f\| \geq c\|T_G V^* f\|$ and $A_1 > 2c$ for all $f \in H$, then $\{Uf_i + Vg_i\}_{i\in I}$ is a K-frame for H.*

Theorem 1.10 ([8]). *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two K-frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively, and $K \in B(H)$. Let T_F and T_G be analysis operators of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, and $T_F^* T_G = 0$. Let $U, V \in B(H)$ with $UK = KU$ and U^* is surjective; then $\{Uf_i + Vg_i\}_{i\in I}$ is a K-frame for H.*

Corollary 1.11 ([8]). *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two K-frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively, and $K \in B(H)$. Let T_F and T_G be analysis operators of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, and $T_F^* T_G = 0$. Let $U, V \in B(H)$, if $UK \in B(H)$, then $\{Uf_i + Vg_i\}_{i\in I}$ is a UK-frame for H.*

2. Main Result

In this section, we present some sufficient conditions for the finite sum of K-frames to be a K^2-frame.

THEOREM 2.1. *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two K-frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively. Let T_F and T_G be analysis operators of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, and $T_F^* T_G = 0$. Let $U, V \in B(H)$, with $\mathcal{R}(K) \subseteq \mathcal{R}(U) + \mathcal{R}(V)$ and $UK = KU$, $VK = KV$. Then $\{Uf_i + Vg_i\}_{i\in I}$ is a K^2-frame for H.*

PROOF. since $\{f_i\}_{i\in I}$ is a Bessel sequence and U is a bounded linear operator, so that $\{Uf_i\}_{i\in I}$ is a Bessel sequence. For the same reason, $\{Vg_i\}_{i\in I}$ is a Bessel sequence and the sum of Bessel sequences is a Bessel sequance. Hence $\{Uf_i + Vg_i\}_{i\in I}$ is a Bessel sequence for H. On the other hand, since $T_F^* T_G = 0$, for all $f \in H$ we have

$$\sum_{i\in I} \langle f, g_i \rangle f_i = 0.$$

Then

$$\begin{aligned}\sum_{i\in I} Re\langle \langle f, Uf_i\rangle, \langle f, Vg_i\rangle\rangle &= \sum_{i\in I} Re\langle \langle U^* f, f_i\rangle, \langle V^* f, g_i\rangle\rangle \\ &= Re\langle \langle U^* f, \sum_{i\in I} \langle V^* f, g_i\rangle f_i\rangle = 0.\end{aligned}$$

Also by Theorem 1.6, $\lambda^2(UU^* + VV^*) \geq KK^*$ for some $\lambda > 0$. Furthermore

$$\begin{aligned}\sum_{i\in I} |\langle f, Uf_i + Vg_i\rangle|^2 &= \sum_{i\in I} |\langle f, Uf_i\rangle|^2 + \sum_{i\in I} |\langle f, Vg_i\rangle|^2 + \sum_{i\in I} 2Re\langle \langle f, Uf_i\rangle, \langle f, Vg_i\rangle\rangle \\ &= \sum_{i\in I} |\langle f, Uf_i\rangle|^2 + \sum_{i\in I} |\langle f, Vg_i\rangle|^2. \\ &\geq A_1 \|K^* U^* f\|^2 + A_2 \|K^* V^* f\|^2 \\ &= A_1 \langle K^* U^* f, K^* U^* f\rangle + A_2 \langle K^* V^* f, K^* V^* f\rangle \\ &= \langle (A_1 UKK^* U^* + A_2 VKK^* V^*) f, f\rangle \\ &\geq \min\{A_1, A_2\} \langle (UKK^* U^* + VKK^* V^*) f, f\rangle \\ &= \min\{A_1, A_2\} \langle (KUU^* K^* + KVV^* K^*) f, f\rangle \\ &= \min\{A_1, A_2\} \langle (K(UU^* + VV^*) K^* f, f\rangle \\ &\geq \frac{\min\{A_1, A_2\}}{\lambda^2} \langle (KK^*) K^* f, K^* f\rangle \\ &= \frac{\min\{A_1, A_2\}}{\lambda^2} \langle K^{*^2} f, K^{*^2} f\rangle \\ &= \frac{\min\{A_1, A_2\}}{\lambda^2} \|K^{*^2} f\|^2.\end{aligned}$$

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