Analytical solution for EGP cylinder under internal pressure using Hyper geometric functions

Vahid Atabakhshian[1], **Hossein Taghipour**[2]

[1]Velayat University, Iranshahr; v.atabakhshian@velayat.ac.ir
[2]Velayat University, Iranshahr; H.taghipour@velayat.ac.ir

Abstract

In this study the hyper-geometric functions such as Whittaker M and Whittaker W, are employed to obtain a closed form solution for exponentially graded piezoelectric (EGP) hollow cylinder subjected to the internal pressure. The material properties except poisson's ratio and thermal conduction coefficient are assumed to be exponentially distributed along radius. The governing equation in polarized form is shown to reduce to a second-order ordinary differential equation (ODE) with variable coefficients for the radial displacement. The electro-thermo-mechanical induced stresses and the electric potential distributions in the EGP cylinder are investigated for the piezo-ceramic PZT_4. It is concluded that the inhomogeneity exponent μ plays a substantial role in radial and circumferential stress distributions. Therefore, results of this investigation can be used for optimum design of EGP thick-walled circular cylinders.

Keywords: PZT4, Analytical Solution, electric field, cylinder

Introduction

It is well known that piezoelectric materials experience mechanical deformations when placed in an electric field, and become electrically polarized under mechanical loads and in fact they exhibit electromechanical coupling. Piezoelectric materials have been used for a long time to make many electromechanical devices. One of the first applications of the piezoelectric effect was an ultrasonic submarine detector developed during the First World War. Other applications include transducers for converting electrical energy to mechanical energy or vice versa, resonators and filters for frequency control and selection for telecommunication and precise timing and synchronization, and acoustic wave sensors. To amend the structure of piezoelectric material and the application of these materials in various environments, functionally graded piezoelectric Material (FGPM) has been created. Mechanical and thermal properties of FGPM are similar to FGM, varying continuously in terms of the spatial coordinate system.

For homogeneous piezoelectric media, Ghorbanpour et al. [1] investigated the stress distribution of dielectric potential fields in piezoelectric hollow spheres. Their results showed that an existing mechanical hoop stress distribution can be neutralized by a suitably applied electric field. Saadatfar and Razavi [2] analyzed the stress in piezoelectric hollow cylinder with thermal gradient. An analytic solution to the axisymmetric problem of an infinitely long, radially polarized, radially orthotropic piezoelectric hollow circular cylinder was developed by Galic and Horgan [3].

For a special inhomogeneous case, Lu et al. [4] derived the exact three-dimensional analytical solutions for a rectangular laminate with piezoelectric layers of exponentially graded material properties along thickness, and under simply supported boundary conditions along two opposite edges. They discussed some properties of the mechanical and electric responses of the FGPM plates under mechanical and/or electrical forces, and they also showed the influence of material gradients by numerical examples based on the exact solutions.

Zhong and Shang [5] presented an exact solution of a simply supported functionally gradient piezoelectric rectangular plate based on three-dimensional electroelasticity theory. The obtained exact solution was valid for arbitrary mechanical and electric loads applied on the upper and lower surfaces of the plate and could serve as a benchmark result to assess other approximate methodologies or as a basis for establishing simplified FGPM plate theories.

The three-dimensional exact solutions of a simply supported FGPM plate/laminate were obtained based on an alternative approach by Lu et al. [6]. They compared the exact solutions with the solutions obtained in [5], and concluded that their exact solutions may be more convenient for further treatments of both analytical and numerical studies.

Pan and Han [7] also presented an exact solution for a multilayered rectangular plate made of anisotropic and functionally graded magneto-electro-elastic materials. The plate is simply supported along its edges, and both mechanical and electric loads are applied on the top surface. Their numerical results showed the influence of the exponential factor, magneto-electro-elastic properties, stacking sequence and loading types on the induced magneto-electro-elastic fields, which should be of interest to the design of smart structures.

The static analysis of a single-layered functionally graded piezoelectric plate, in both sensor and actuator configuration was investigated by Brischetto and Carrera [8]. They compared refined theories with classical ones, to demonstrate the effectiveness of these theories in the case of FGPMs.

Recently, Li et al. [9] analyzed an axisymmetric electro-elastic problem of hollow radially polarized piezoceramic cylinders made of FGMs as sensors and actuators. For two typical cases of sensors and actuators, the response of the radial and circumferential stresses as well as the electric potential was shown graphically by them. Their derived results for the distribution of hoop and radial stresses as well as the electric response are useful in designing hollow cylindrical FG piezoelectric sensors/ actuators.

The elastic and piezoelectric properties vary as a power-function along radius, Khoshgoftar et al. [10] studied thermo-piezoelectric behavior of a thick-walled cylinder made of FGMs subjected to a temperature gradient and inner and outer pressures.

Babaei and Chen [11] presented the analytical solution for a radially piezoelectric functionally graded rotating hollow shaft. The analytical solution of a functionally graded piezo-thermo-elastic hollow cylinder was presented by Chen and Shi [12]. They assumed that only the piezoelectric coefficient was varying quadratically in the radial direction while the other material parameters were assumed to be constants. They neglected other inhomogeneity parameters such as thermal conduction coefficient and modulus of elasticity. Because of this incompleteness and more accordance of exponential form of properties with new practical model, the effect of the entire material inhomogeneity on the piezo-thermo-electro-elastic behavior of an EGP hollow cylinder is studied in the present paper.

In this study a thick-walled cylinder of radially polarized anisotropic piezoelectric material, e.g., PZT_4 is assumed. The cylinder is subjected to mechanical and thermal loads, together with a potential difference induced by electrodes attached to the inner and outer surfaces (Fig. 1). All the mechanical, thermal and piezoelectric properties of the EGP hollow cylinder, except Poisson's ratio, are assumed to be radial dependent and they are expressed as exponential functions of The governing equations of exponentially graded piezoelectric cylindrically structure in radially polarized form are reduced to a second–order inhomogeneous differential equation in terms of displacement with variable coefficients and is solved using hyper-geometric functions and Simpson's integrating method. Finally, numerical results and the associated data are presented for four different sets of boundary conditions to illustrate electro-elastic stresses, electric potential, radial displacement and thermal distributions for piezo-ceramic, PZT_4.

3 Basic formulation

In this section, the general differential equation of a hollow cylinder with inhomogeneous material properties is obtained. The material properties are exponentially variable along the radial direction as follows:

$$c_{ij}(r)=c_{ij0}e^{[\mu\frac{r-a}{b-a}]},\ e_{1i}(r)=e_{1i0}e^{[\mu\frac{r-a}{b-a}]},\ g_{11}(r)=g_{110}e^{[\mu\frac{r-a}{b-a}]},$$
$$p_{11}(r)=p_{110}e^{[\mu\frac{r-a}{b-a}]},\ \rho(r)=\rho_0 e^{[\mu\frac{r-a}{b-a}]}, \tag{1}$$

The constitutive relations of stresses and displacement in radially polarized piezoelectric cylinder and the component of radial electric displacement vector are written as [14]

$$\sigma_r=c_{11}\frac{\partial u}{\partial r}+c_{12}\frac{u}{r}+e_{11}\frac{\partial\phi(r)}{\partial r}-\lambda_1 T(r), \tag{2}$$

$$\sigma_\theta=c_{12}\frac{\partial u}{\partial r}+c_{22}\frac{u}{r}+e_{12}\frac{\partial\phi(r)}{\partial r}-\lambda_2 T(r), \tag{3}$$

$$D_r=e_{11}\frac{\partial u}{\partial r}+e_{12}\frac{u}{r}-g_{11}\frac{\partial\phi(r)}{\partial r}+p_{11}T(r), \tag{4}$$

where σ_r,σ_θ are radial and circumferential stresses, D_r is the radial electric displacement and the thermal modulus λ_1,λ_2 are given by

$$\lambda_1=c_{11}\alpha_1+c_{12}\alpha_2, \tag{5}$$

$$\lambda_2=c_{12}\alpha_1+c_{22}\alpha_2. \tag{6}$$

The equation of equilibrium and the charge equation of cylinder are expressed as [15 and 16]

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\rho r\omega^2=0, \tag{7}$$

$$\frac{\partial D_r}{\partial r}+\frac{D_r}{r}=0. \tag{8}$$

Solution procedure

By solving the differential equation (7) the electrical displacement is obtained as

$$D_r(r)=\frac{A_1}{r}, \tag{9}$$

Substituting (9) into (4) the following equation is obtained for potential gradient as

$$\frac{\partial\phi(r)}{\partial r}=\frac{1}{g_{110}}[e_{110}\frac{\partial u}{\partial r}+e_{120}\frac{u}{r}+p_{110}T(r)-\frac{A_1}{r}e^{-[\mu\frac{r-a}{b-a}]}], \tag{10}$$

where A_1 is a constant.

Substituting (10) into (2) and (3), the radial and hoop stresses can be rewritten as

$$\sigma_r=e^{[\xi(r-a)]}[w_1\frac{\partial u}{\partial r}+w_2\frac{u}{r}+w_3T(r)]-\frac{w_4A_1}{r}, \tag{11}$$

$$\sigma_\theta=e^{[\xi(r-a)]}[w_2\frac{\partial u}{\partial r}+w_5\frac{u}{r}+w_6T(r)]-\frac{w_7A_1}{r}. \tag{12}$$

Substituting (11) and (12) into (8) the governing differential equation for the problem in terms of radial displacement and temperature distribution $T(r)$ is derived as

$$r^2\frac{\partial^2u}{\partial r^2}+(1+\xi r)r\frac{\partial u}{\partial r}+(\frac{\xi w_2r-w_5}{w_1})u=(-\frac{\xi w_3r^2+r(w_3-w_6)}{w_1})T(r)$$
$$-\frac{w_3}{w_1}r^2\frac{\partial T(r)}{\partial r}-\frac{\rho_0\omega^2}{w_1}r^3-\frac{w_7A_1}{w_1}e^{[-\xi(r-a)]}, \tag{13}$$

Simplifying equation (13), yields

$$r^2\frac{\partial^2u}{\partial r^2}+(1+\xi r)r\frac{\partial u}{\partial r}+(\frac{\xi w_2r-w_5}{w_1})u=F(r). \tag{14}$$

The inhomogeneous term on the right-hand-side of (14) is denoted as follows

$$F(r)=-D_2(\frac{\xi w_3r^2+r(w_3-w_6)}{w_1})-D_1(\frac{\xi w_3r+w_3-w_6-w_3\beta}{w_1})r^{1-\beta}-\frac{\rho_0\omega^2}{w_1}r^3$$
$$-\frac{w_7A_1}{w_1}e^{[-\xi(r-a)]}. \tag{15}$$

The general solution of (14) is assumed to be in the following form:

$$u(r)=C_1P(r)+C_2Q(r)+R(r), \quad (16)$$

where C_1 and C_2 are arbitrary integration constants, $P(r)$ and $Q(r)$ are homogenous solutions, and $R(r)$ is the particular solution.
The particular solution can be developed by employing the variation of parameter methods
The corresponding partial differential equations (14) are solved analytically by Maple software and the homogenous and particular solution are obtained using hypergeometric functions.

Numerical results and discussion

In this study the cylinder is considered to be composed of PZT4 and subjected to the internal pressure and free for all other boundary conditions.
The following data for geometry, material properties, boundary and loading conditions are considered

$a=0.1m$, $b=0.2m$, $\omega=400\,rad/s$, $P_a=3e7(Pa)$,

$\phi_a=1000(W/A)$, $T_0=20^\circ C$, $\beta=1$ $\mu=-2,-1,0,1 \text{ and } 2$

The following results are reported based on the following normalized variables

$$\zeta=\frac{r-a}{b-a},\sigma_i^*=\frac{\sigma_i}{P_a}(i=r,\theta),U^*=\frac{u(r)}{a},\Phi^*=\frac{\phi}{\phi_a}\quad T^*=\frac{T(r)}{T_0}$$

In Figures 1 and 2, results are shown for, where internal pressure is applied. Fig.1 depicts the distribution of radial stress along the radius for different values of μ. It is seen from the figure that the radial stresses satisfy the mechanical boundary conditions. The distributions of the hoop stresses for different values of μ are displayed in Fig.2. In these figures we can easily observe a reference stress point at $\zeta=0.2$ in which the hoop stress is identical for all material properties except for $\mu=1$. Furthermore, this figure shows that the hoop stresses are increasing at the outer surface and decreasing at the inner surface with increasing the material inhomogeneity exponent μ.
Although for the first case no electric potential is imposed however an induced electric potential distribution satisfying the zero potential boundary condition is obtained in which the minimum potential distribution belongs to the material identified by $\mu=-2$ and the maximum distribution belongs to $\mu=2$. It is concluded from Fig. 3 that the material inhomogeneity exponent μ significantly affects the induced electric potential.
The radial displacement distributions for different values of the inhomogeneity material μ are demonstrated in Fig. 4. It is seen from this figure that the radial displacement decreases with increasing μ and its maximum value is located at the inner radius of the cylinder.

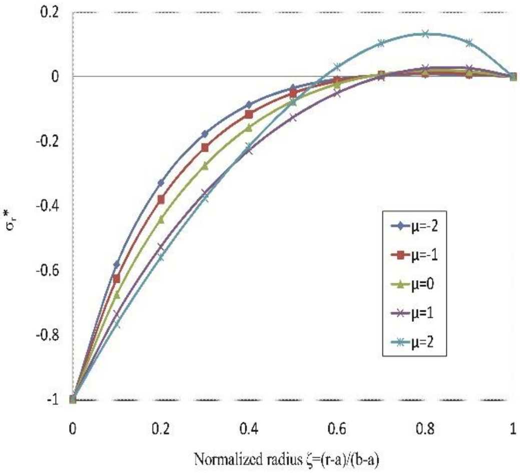

Figure 1. Distribution of radial stress for different values of μ

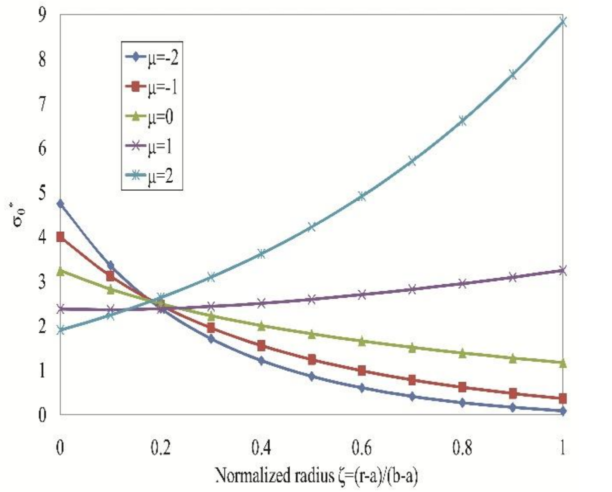

Figure 2. Distribution of circumferential stress for different values of μ

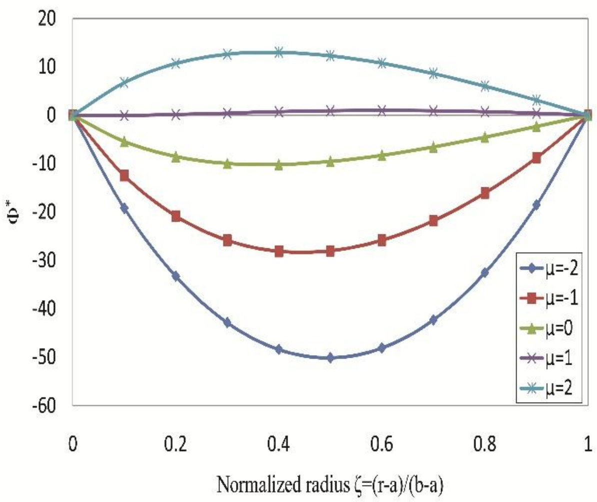

Figure 3. Distribution of electric potential for different values of μ

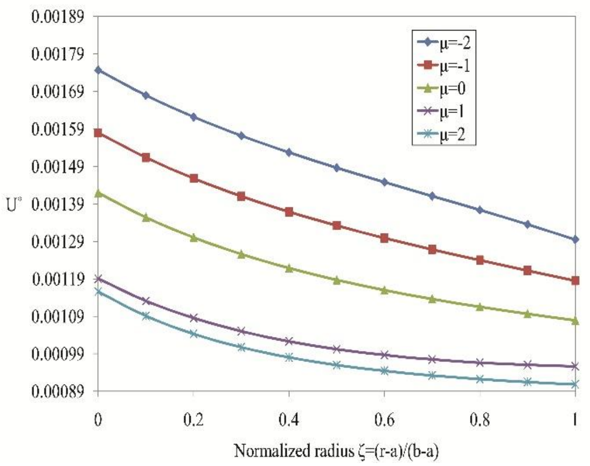

Figure 4. Distribution of radial displacement for different values of μ

Conclusions

Using the hyper-geometric function, closed form solution for EGP hollow cylinder, is presented with four different sets of boundary conditions. The analytical solution of homogeneous piezoelectric hollow cylinder is fully covered by setting the material inhomogeneity , our results for the homogeneous PZT_4 hollow cylinder are almost the same as those by Galic and Horgan [19]. The correctness of the present solution is then verified in this respect.

It is concluded from the present result that the inhomogeneity exponent significantly affects the radial and hoop stress distributions. This implies that the electro-thermo-mechanical fields in EGP cylindrical structures can be optimized for specific applications by selecting a suitable inhomogeneity parameter. Moreover, the inhomogeneous constants presented in the present study are useful parameters from a design point of view in that they can be tailored for special applications to control the distributions of electro-thermo-elastic stresses. The technological implications of this study are significant, e.g. the amount of hoop stress resulting from thermo-mechanical loads in an EGP hollow cylinder can be reduced or neutralized by a suitably applied electric potential field and material inhomogeneity. Hence forward, our results can help engineers to design a piezoelectric pressure vessel made of EGP material with an optimum distribution of stresses.

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